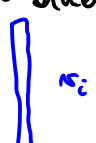





AN INTRO TO TENSOR NETWORKS

vectors $v \in \mathbb{R}^d$
(1st order)



matrices $M \in \mathbb{R}^{m \times m}$
(2nd order)


tensor $T \in \mathbb{R}^{d_1 \times d_2 \times d_3}$
(3rd order)


In TENSOR NETWORKS : ① Nodes \equiv Tensor
Number of "legs" \equiv order of a tensor

$v \in \mathbb{R}^d$


$M \in \mathbb{R}^{m \times m}$


$T \in \mathbb{R}^{d_1 \times d_2 \times d_3}$


② EDGES \equiv CONTRACTIONS

$AB = \overset{m \times m}{A} \overset{m \times p}{B} \in \mathbb{R}^{m \times p}$

$\left(\overset{m \times m}{A} \overset{m \times p}{B} \right)_{ij} = \sum_{k=1}^m A_{ik} B_{kj}$ for all $i=1 \dots m$
 $j=1 \dots p$

$u, v \in \mathbb{R}^d \rightsquigarrow \underbrace{u \text{---} v}_{\in \mathbb{R}} = \sum_{i=1}^d u_i v_i = u^T v = \langle u, v \rangle$
inner product

$u \in \mathbb{R}^m, v \in \mathbb{R}^m \rightsquigarrow \underbrace{u \text{---} v}_{\in \mathbb{R}^{m \times m}}, \left(\begin{matrix} u & v \\ | & | \\ 1 & 1 \end{matrix} \right)_{ij} = u_i v_j$
 $\Rightarrow \underbrace{u \text{---} v}_{\in \mathbb{R}^{m \times m}} = u v^T = u \otimes v$
outer product

$A \in \mathbb{R}^{m \times m} \rightsquigarrow \underbrace{\overset{m}{A} \text{---} \overset{m}{A}}_{\in \mathbb{R}} = \sum_{i=1}^m A_{ii} = \text{Tr}(A)$

Proposition: $\text{Tr}(AB) = \text{Tr}(BA)$,

$\text{Tr}(ABC) = \text{Tr}(CAB) = \text{Tr}(BCA)$

proof: $\text{Tr}(AB) = \underbrace{\overset{m}{A} \text{---} \overset{m}{B}}_{\in \mathbb{R}} = \overset{m}{\begin{pmatrix} A \\ B \end{pmatrix}} = \underbrace{\overset{m}{B} \text{---} \overset{m}{A}}_{\in \mathbb{R}} = \text{Tr}(BA)$

$\text{Tr}(ABC) = \underbrace{A \text{---} B \text{---} C}_{\in \mathbb{R}} = \overset{m}{\begin{pmatrix} A \\ B \\ C \end{pmatrix}} = \text{Tr}(BCA) = \text{Tr}(CAB)$

• COPY TENSOR

$$\mathbb{R}^{m \times m \times m} \Rightarrow \begin{matrix} m \\ \text{---} \\ m \\ \text{---} \\ m \end{matrix}$$

$$\left(\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \right)_{ijk} = \begin{cases} 1 & \text{if } i=j=k \\ 0 & \text{otherwise} \end{cases} = \delta_{ijk}$$

$$\left(\begin{matrix} \text{---} \\ \text{---} \\ \dots \\ \text{---} \end{matrix} \right)_{i_1 i_2 \dots i_n} = \begin{cases} 1 & \text{if } i_1 = i_2 = \dots = i_n \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} = \mathbf{I} \quad \text{and} \quad \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Properties: $\left(\begin{matrix} A \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} \right)_i = \sum_{j=1}^m \sum_{k=1}^m A_{jk} \delta_{ijk} = A_{ii}$

$\in \mathbb{R}^m$

$$\Rightarrow \left(\begin{matrix} A \\ \text{---} \end{matrix} \right) = \text{diag}(A)$$

$$\left(\begin{matrix} \nu \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix} \right)_{ij} = \sum_{k=1}^d \nu_k \delta_{ijk} = \begin{cases} 0 & \text{if } i \neq j \\ \nu_i & \text{if } i = j \end{cases}$$

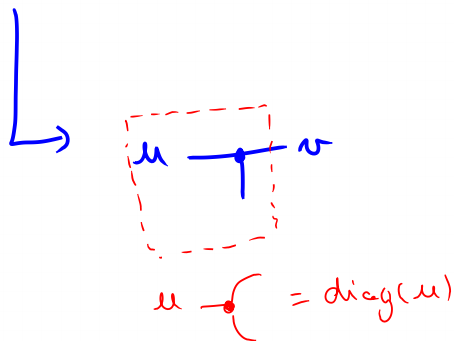
$\in \mathbb{R}^{d \times d}$

$$\left(\begin{matrix} \mu & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \nu \end{matrix} \right)_i = \sum_{j,k} \mu_j \nu_k \delta_{ijk} = \mu_i \nu_i$$

$\in \mathbb{R}^d$

$\mu \text{---} \nu = \mu \otimes \nu$ Hadamard / Component wise product.

Obs: $\mu \otimes \nu = \text{diag}(\mu) \nu = \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \ddots \\ & & & \mu_d \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix}$



RESHAPING

Vectorization:

$$\mathbb{R}^{m \times m} \ni A = \begin{pmatrix} | & & | \\ a_1 & \dots & a_m \\ | & & | \end{pmatrix} \cong \text{vec}(A) = \begin{bmatrix} a_1 \\ \vdots \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^{mm}$$

In TN: $m \underbrace{(A)}_m \cong \underbrace{A}_{m|m}$

Tensors to matrices:

$$\mathbb{R}^{d_1 \times d_2 \times d_3} \ni \underbrace{\quad}_T \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix} \rightarrow \underbrace{\quad}_{d_3} \begin{matrix} d_1 \\ d_2 \end{matrix} \in \mathbb{R}^{d_1 \times d_2 d_3}$$

$$\rightarrow \underbrace{\quad}_{d_3} \begin{matrix} d_2 \\ d_1 \end{matrix} \in \mathbb{R}^{d_2 \times d_1 d_3}$$

$$\rightarrow \underbrace{\quad}_{d_1 d_2 d_3} \in \mathbb{R}^{d_1 d_2 d_3}$$

KRONECKER PRODUCT

$\hookrightarrow A \in \mathbb{R}^{m \times m}$

$B \in \mathbb{R}^{p \times q}$

$$\mathbb{R}^{mp \times mq} \ni A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & \dots & \dots & a_{mm}B \end{bmatrix}$$

TN: $\left(\underbrace{m \underbrace{(A)}_m \quad p \underbrace{(B)}_q}_{\mathbb{R}^{m \times m \times p \times q}} \right)_{ijke} = A_{ij} B_{ke}$

Reshape it to matrix

$$\mathbb{R}^{mm \times pq} \ni \underbrace{A}_{m|m} \quad \underbrace{B}_{p|q} = \text{vec}(A) \text{vec}(B)^T$$

$\text{vec}(A) \quad \text{vec}(B)$

$$\mathbb{R}^{mp \times mq} \ni \underbrace{\quad}_{p \quad q} \begin{matrix} m & A & m \\ \hline & B & \hline \end{matrix} = A \otimes B$$

\Rightarrow KRONECKER PRODUCT IS A TENSOR

Fluxed product property.

$$\begin{aligned}
 (A \otimes C) (B \otimes D) &= \begin{array}{c} \overbrace{A}^m \text{---} \overbrace{C}^q \\ \underbrace{\hspace{10em}}_{A \otimes C} \end{array} \begin{array}{c} \overbrace{B}^p \text{---} \overbrace{D}^s \\ \underbrace{\hspace{10em}}_{B \otimes D} \end{array} \\
 \begin{array}{c} \uparrow \quad \uparrow \\ m \times m \quad p \times q \\ \hline m \times p \times m \times q \\ \hline m \times p \times n \times q \\ \hline m \times p \times n \times s \end{array} &= \begin{array}{c} \overbrace{A \text{---} B}^m \text{---} \overbrace{C \text{---} D}^q \\ \underbrace{\hspace{10em}}_{A \otimes C} \end{array} \begin{array}{c} \overbrace{B}^p \text{---} \overbrace{D}^s \\ \underbrace{\hspace{10em}}_{B \otimes D} \end{array} \\
 &= \begin{array}{c} \overbrace{A \text{---} B}^m \text{---} \overbrace{C \text{---} D}^q \\ \underbrace{\hspace{10em}}_{A \otimes C} \end{array} \begin{array}{c} \overbrace{B}^p \text{---} \overbrace{D}^s \\ \underbrace{\hspace{10em}}_{B \otimes D} \end{array} = AB \otimes CD
 \end{aligned}$$

GRADIENTS OF TENSOR NETWORKS

If a tensor T defined by a TN where a tensor G appears only once, then the Jacobian of T w.r.t. G is obtained by removing G from the TN.

$$\frac{\partial A x}{\partial x} = \frac{\partial \text{---} A \text{---} x}{\partial x} = \text{---} A \text{---}$$

$$\frac{\partial x^T A x}{\partial A} = \frac{\partial x \text{---} A \text{---} x}{\partial A} = x \text{---} \text{---} x = x x^T$$

$$\begin{aligned}
 \frac{\partial A x}{\partial A} &= \frac{\partial \text{---} A \text{---} x}{\partial A} = \begin{array}{c} \overset{m}{\uparrow} \quad \overset{m}{\uparrow} \\ \text{---} \quad \text{---} \\ \text{---} \end{array} \\
 &\approx \text{---} I \text{---} x
 \end{aligned}$$