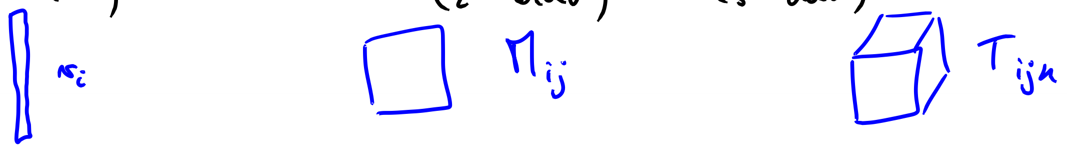
An Intro to Tensor Networks

vectors $v \in \mathbb{R}^d$ (1st order) matrices $M \in \mathbb{R}^{m\times n}$ (2nd order) tensor $T \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ (3rd order)



In Tensor Networks : ① Node $\equiv$ Tensor

Number of "legs" $\equiv$ order of a tensor

$v \in \mathbb{R}^d$ $M \in \mathbb{R}^{m\times n}$ $T \in \mathbb{R}^{d_1\times d_2 \times d_3}$

② Edges $\equiv$ Contractions

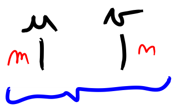
$$\overset{m\times n}{A}\overset{n\times p}{B} = \overset{m}{—} A \overset{n}{—}\!\!\overset{n}{—} B \overset{p}{—} \in \mathbb{R}^{m\times p}$$

$$\left(\overset{m}{—} A \overset{n}{—}\!\!\overset{n}{—} B \overset{p}{—}\right)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \quad \text{for all } \begin{matrix} i = 1\dots m \\ j = 1 \dots p \end{matrix}$$

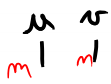
$$u, v \in \mathbb{R}^d \rightsquigarrow \underbrace{u \overset{d}{—\!—} v}_{\in \mathbb{R}} = \sum_{i=1}^d u_i v_i = u^T v = \langle u, v\rangle$$

↑ inner product

$u \in \mathbb{R}^m, v \in \mathbb{R}^n \rightsquigarrow$



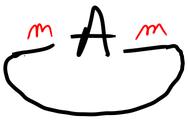
$\in \mathbb{R}^{m\times n}$, $\left(\begin{matrix} u & v \\ | & | \end{matrix}\right)_{ij} = u_i v_j$



$\Rightarrow$ $= u v^T = u \circ v$

↑ outer product

$A \in \mathbb{R}^{m\times m} \rightsquigarrow$



$= \sum_{i=1}^{m} A_{ii} = Tr(A)$

$\in \mathbb{R}$

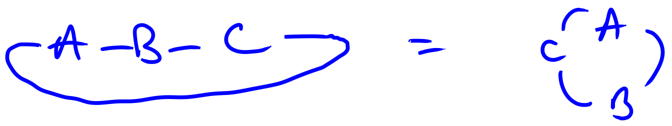
Proposition : $Tr(AB) = Tr(BA)$,

$Tr(ABC) = Tr(CAB) = Tr(BCA)$

proof : $Tr(AB) =$

$= Tr(BA)$

$Tr(ABC) =$



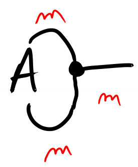
$= Tr(BCA)$
$= Tr(CAB)$

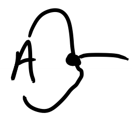
- Copy Tensor

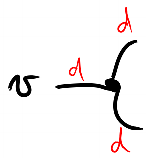
$\mathbb{R}^{m\times m\times m} \ni$  $\left(\right.$  $\left.\right)_{ijk} = \begin{cases} 1 & \text{if } i=j=k \\ 0 & \text{otherwise} \end{cases} = \delta_{ijk}$

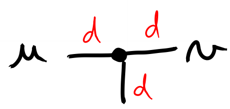
$\left(\right.$  $\left.\right)_{i_1 i_2 \dots i_N} = \begin{cases} 1 & \text{if } i_1 = i_2 = \dots = i_N \\ 0 & \text{o.w.} \end{cases}$

 $= I$ and  $= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

Properties: $\underbrace{\left(\right.}_{\in \mathbb{R}^m}$  $\left.\right)_i = \sum_{j=1}^{m} \sum_{k=1}^{m} A_{jk}\, \underset{i=j=k}{\delta_{ijk}} = A_{ii}$

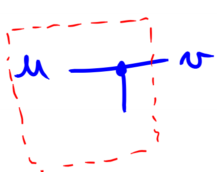
$\Rightarrow$  $= \text{diag}(A)$

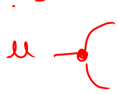
$\underbrace{\left(\right.}_{\in \mathbb{R}^{d\times d}}$  $\left.\right)_{ij} = \sum_{k=1}^{d} v_k \delta_{ijk} = \begin{cases} 0 & \text{if } i \neq j \\ v_i & \text{if } i = j \end{cases}$

$\underbrace{\left(\right.}_{\in \mathbb{R}^{d}}$  $\left.\right)_i = \sum_{j,k} u_j v_k \delta_{ijk} = u_i v_i$

 $= u \circledast v$ Hadamard / Component wise product.

Obs: $u \circledast v = \text{diag}(u)\, v = \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_d \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}$



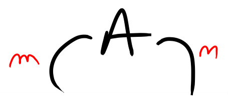
 $= \text{diag}(u)$

RESHAPING

Vectorization:

$$\mathbb{R}^{m\times m} \ni A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{pmatrix} \cong \mathrm{vec}(A) = \begin{bmatrix} | \\ a_1 \\ | \\ | \\ a_2 \\ | \\ \vdots \\ | \\ a_m \\ | \end{bmatrix} \in \mathbb{R}^{mm}$$

In TN:

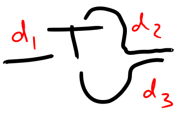


$\cong$

Tensors to matrices:

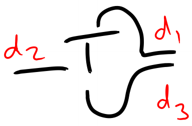
$\mathbb{R}^{d_1\times d_2\times d_3} \ni$

$\rightarrow$



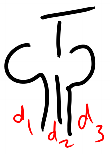
$\in \mathbb{R}^{d_1\times d_2 d_3}$

$\rightarrow$



$\in \mathbb{R}^{d_2\times d_1 d_3}$

$\rightarrow$



$\in \mathbb{R}^{d_1 d_2 d_3}$

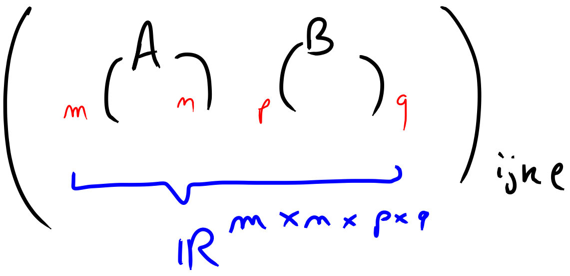
KRONECKER PRODUCT

$\hookrightarrow A \in \mathbb{R}^{m\times m}, \; B \in \mathbb{R}^{p\times q}$

$a_{11}b_{11} \;\; a_{11}b_{12} \cdots$

$$\mathbb{R}^{mp\times mq} \ni A \otimes B = \begin{bmatrix} \boxed{a_{11}B} & a_{12}B & \cdots & a_{1m}B \\ \vdots & & \ddots & \vdots \\ a_{m1}B & \cdots & \cdots & a_{mm}B \end{bmatrix}$$

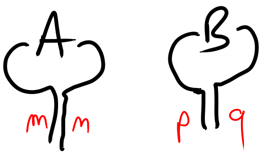
TN:



$= A_{ij} B_{kl}$

Reshape it to matrix

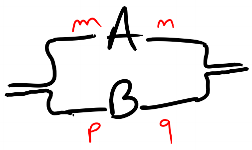
$\mathbb{R}^{mm\times pq} \ni$



$= \mathrm{vec}(A)\,\mathrm{vec}(B)^T$

vec(A)   vec(B)

$\mathbb{R}^{mp\times mq} \ni$



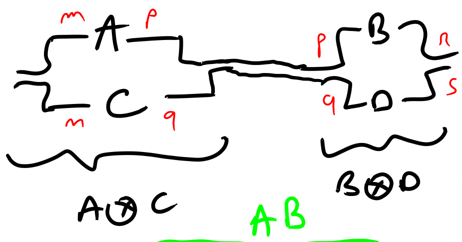
$= A \otimes B$

$\Rightarrow$ KRONECKER PRODUCT is A TENSOR

Mixed product property.

$$\underbrace{(\underset{m\times m}{A}\otimes \underset{p\times q}{C})}_{mp\times mq}\underbrace{(\underset{m\times n}{B}\otimes \underset{q\times s}{D})}_{mq\times ns} \quad (mp\times ns)$$

=



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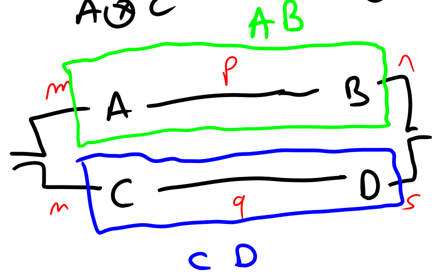

$= AB\otimes CD$

GRADIENTS OF TENSOR NETWORKS

If A tensor T defined by a TN where a tensor G appears only once, then the Jacobian of T w.r.t. G is obtained by removing G from the TN.

$$\frac{\partial Ax}{\partial x} = \frac{\partial \; -A-x}{\partial x} = -A-$$

$$\frac{\partial x^T A x}{\partial A} = \frac{\partial \; x-A-x}{\partial A} = x- \quad -x$$

$$= xx^T$$

$$\frac{\partial \; Ax}{\partial A} = \frac{\partial \; \overset{m}{-}A\overset{m}{-}x}{\partial A} = \underset{I}{\overset{m}{-}} \quad \underset{x}{\overset{m}{-}x}$$

$$\simeq I \circ x$$